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Irreducible representations of Dirac algebra for a constrained system

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Abstract
All possible irreducible representations of the Dirac algebra for a particle constrained to move on a D-dimensional manifold $f(x) = 0$ are explicitly constructed in terms of canonical operators $\hat{x}_\alpha, \hat{\pi}_\alpha$ $(\alpha = 1, 2, \ldots, D+1)$ in $\mathbb{R}^{D+1}$ by assuming that the manifold, which is embedded in $\mathbb{R}^{D+1}$, is diffeomorphic to S^D. It is shown that for $D = 1$ any irreducible representation is uniquely specified by a real parameter α belonging to $[0, 1)$, while for $D \geqslant 2$ the irreducible representation is unique. The explicit form of inner products in $\hat{x}_\alpha$-diagonal representation is given with the help of auxiliary wavefunctions on $\mathbb{R}^{D+1}$, provided that they satisfy certain boundary conditions on the manifold. Applying it we further examine the hermiticity property of the fundamental operators of the Dirac algebra.

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1. Introduction

About 50 years ago Dirac [1] proposed a canonical formalism for constrained systems. On the basis of his approach we will, in the present paper, make a study of some rigorous treatments for a quantum-mechanical system constrained to move on a D-dimensional manifold, which is embedded in the flat space $\mathbb{R}^{D+1}$ with coordinates x_α $(\alpha = 1, 2, \ldots, D+1)$. The classical Hamiltonian in $\mathbb{R}^{D+1}$ is assumed to be

$$H = \frac{1}{2}p^2 + V(x) \tag{1.1}$$

where $p^2 = p_\alpha p_\alpha$ and repeated Greek indices indicate the summation over $1, 2, \ldots, D+1$. The D-dimensional smooth manifold on which the system is constrained will be denoted as

$$f(x) = 0 \tag{1.2}$$

with a real function $f(x)$. We require that the manifold be diffeomorphic to S^D.

Equation (1.2) is the so-called primary constraint. According to the prescription by Dirac the consistency of (1.2) under time development leads us to the secondary constraint

$$p_\alpha f_{,\alpha}(x) = 0 \tag{1.3}$$

where and in what follows $f_{,\alpha}(x) \equiv \partial_\alpha f(x)$, $f_{,\alpha\beta}(x) \equiv \partial_\alpha \partial_\beta f(x)$. Then the Dirac brackets (denoted as $[\ ,\ ]^*$) for canonical variables in classical mechanics are given by

$$[x_\alpha, x_\beta]^* = 0 \tag{1.4}$$

$$[x_\alpha, p_\beta]^* = \Lambda_{\alpha\beta}(x) \tag{1.5}$$

$$[p_\alpha, p_\beta]^* = \frac{-1}{R^2(x)}(f_{,\alpha}(x) f_{,\beta\gamma}(x) - f_{,\beta}(x) f_{,\alpha\gamma}(x)) p_\gamma \tag{1.6}$$

where $\Lambda_{\alpha\beta}(x)$ stands for the projection

$$\Lambda_{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{f_{,\alpha}(x) f_{,\beta}(x)}{R^2(x)} \qquad \text{with} \quad R^2(x) = f_{,\alpha}(x) f_{,\alpha}(x). \tag{1.7}$$

Thus it is considered that a quantum-mechanical version of the above relations will be of the form

$$f(\hat{x}) = 0 \tag{1.8}$$

$$\{\hat{p}_\alpha, f_{,\alpha}(\hat{x})\} = 0 \tag{1.9}$$

$$[\hat{x}_\alpha, \hat{x}_\beta] = 0 \tag{1.10}$$

$$[\hat{x}_\alpha, \hat{p}_\beta] = \mathrm{i}\hbar \Lambda_{\alpha\beta}(\hat{x}) \tag{1.11}$$

$$[\hat{p}_\alpha, \hat{p}_\beta] = -\frac{\mathrm{i}\hbar}{2}\left\{\frac{1}{R^2(\hat{x})}(f_{,\alpha}(\hat{x}) f_{,\beta\gamma}(\hat{x}) - f_{,\beta}(\hat{x}) f_{,\alpha\gamma}(\hat{x})), \hat{p}_\gamma\right\} \tag{1.12}$$

where $\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$ and the symbol hat $\hat{}$ indicates an operator. Needless to say the Hamiltonian in this case is $\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{x})$. Since the manifold $f(x) = 0$ is diffeomorphic to S^D, it is obvious that

$$R^2(x) \neq 0 \tag{1.13}$$

in the neighbourhood of $f(x) = 0$.

With direct calculation it is shown that the commutation relations (1.10)–(1.12) are compatible with the constraints (1.8) and (1.9). We call the set of relations (1.8)–(1.12) the Dirac algebra on $f(x) = 0$. Unlike the case of classical mechanics we are required to determine irreducible representations of this algebra, because the structure of our theory inevitably depends on the representation.

To this end, in section 2 a study is made of a connection between two irreducible representations of Dirac algebras defined respectively on $f(x) = 0$ and $g(x) = 0$ which are both diffeomorphic to S^D.

Based on this argument, in section 3, all possible irreducible representations of the Dirac algebra on $f(x) = 0$ are completely determined by identifying the manifold $g(x) = 0$ with S^D. Corresponding to each irreducible representation the explicit form of $\hat{p}_\alpha$ is shown to be described in terms of canonical variables on $\mathbb{R}^{D+1}$.

In section 4 with the help of auxiliary L^2-functions on $\mathbb{R}^{D+1}$ we try to rewrite inner products of state vectors in a form convenient for practical use. With the aid of this the Hermitian property of $\hat{p}_\alpha$ on the irreducible representation space is examined.

Section 5 is devoted to some additional remarks.

2. Two Dirac algebras

We will examine a relation between representations of two Dirac algebras defined respectively on the manifolds $f(x) = 0$ and $g(x) = 0$, which are both diffeomorphic to S^D. As in the case of $f(x) = 0$ we can write the Dirac algebra on $g(x) = 0$ assuming the Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{x})$. It takes the following form:

$$g(\hat{x}) = 0 \tag{2.1}$$

$$\{g_{,\alpha}(\hat{x}), \hat{p}_\alpha\} = 0 \tag{2.2}$$

$$[\hat{x}_\alpha, \hat{x}_\beta] = 0 \tag{2.3}$$

$$[\hat{x}_\alpha, \hat{p}_\beta] = \mathrm{i}\hbar\Lambda'_{\alpha\beta}(\hat{x}) \tag{2.4}$$

$$[\hat{p}_\alpha, \hat{p}_\beta] = -\frac{\mathrm{i}\hbar}{2}\left\{\frac{1}{R'^2(\hat{x})}(g_{,\alpha}(\hat{x})g_{,\beta\gamma}(\hat{x}) - g_{,\beta}(\hat{x})g_{,\alpha\gamma}(\hat{x})), \hat{p}_\gamma\right\} \tag{2.5}$$

where

$$\Lambda'_{\alpha\beta}(\hat{x}) = \delta_{\alpha\beta} - \frac{g_{,\alpha}(\hat{x})g_{,\beta}(\hat{x})}{R'^2(\hat{x})} \qquad \text{and} \qquad R'^2(\hat{x}) = g_{,\alpha}(\hat{x})g_{,\alpha}(\hat{x}). \tag{2.6}$$

Since the manifolds $f(x) = 0$ and $g(x) = 0$ are connected by diffeomorphism, we may write

$$g(x') = f(x) \tag{2.7}$$

with

$$x'_\alpha = x'_\alpha(x) \qquad \text{(or equivalently } x_\alpha = x_\alpha(x')). \tag{2.8}$$

With these preparations we introduce the transformation such that

$$\begin{cases} \hat{x}'_\alpha = \hat{x}'_\alpha(\hat{x}) \\ \hat{p}'_\alpha = \frac{1}{2}\{(\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\alpha\beta}, \hat{p}_\beta\} \end{cases} \tag{2.9}$$

where $\Lambda'(\hat{x}')$ and $[\partial\hat{x}/\partial\hat{x}']$ stand for $(D+1)\times(D+1)$-matrices whose (α,β)-elements are given by $\Lambda'_{\alpha\beta}(\hat{x}')$ and $\partial\hat{x}_\beta/\partial\hat{x}'_\alpha$, respectively. Similarly we define the $(D+1)\times(D+1)$-matrices $\Lambda(\hat{x})$ and $[\partial\hat{x}'/\partial\hat{x}]$ with $(\Lambda(\hat{x}))_{\alpha\beta} = \Lambda_{\alpha\beta}(\hat{x})$ and $[\partial\hat{x}'/\partial\hat{x}]_{\alpha\beta} = \partial\hat{x}'_\beta/\partial\hat{x}_\alpha$, respectively. Equations (2.9) provide a variable transformation $(\hat{x},\hat{p}) \to (\hat{x}',\hat{p}')$ on the representation space of the Dirac algebra on $f(x) = 0$. Then there holds the following:

Theorem 1. *Given $\hat{x}_\alpha$ and $\hat{p}_\alpha$ that satisfy the Dirac algebra on $f(x) = 0$ then the operators $\hat{x}'_\alpha$ and $\hat{p}'_\alpha$ defined by (2.9) satisfy the Dirac algebra on $g(x) = 0$ for an arbitrary diffeomorphism obeying (2.7).*

Before going to a proof of this theorem we make the following remarks:

1. *Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be operators. If $[[\hat{C},\hat{A}],\hat{B}] = 0$, then*

$$\{\hat{A},\{\hat{B},\hat{C}\}\} = \{\{\hat{A},\hat{B}\},\hat{C}\}. \tag{2.10}$$

Thus if further $[\hat{A},\hat{B}] = 0$, we have

$$\tfrac{1}{2}\{\hat{A},\{\hat{B},\hat{C}\}\} = \{\hat{A}\hat{B},\hat{C}\}. \tag{2.11}$$

The proof of this statement is not difficult and will be omitted here.

2. *There holds true the identity*

$$\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}]\Lambda'(\hat{x}') = \Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}]. \tag{2.12}$$

Proof. Inserting $\Lambda'_{\gamma\beta}(\hat{x}')$, which is defined by (2.6) with the replacement $\hat{x} \to \hat{x}'$, into the left-hand side (lhs) of the above we find

$$\begin{aligned}(\alpha, \beta)\text{-element of lhs} &= (\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}])_{\alpha\beta} - \frac{1}{R'^2(\hat{x}')}\Lambda_{\alpha\rho}(\hat{x})\frac{\partial\hat{x}'_\gamma}{\partial\hat{x}_\rho}g_{,\gamma}(\hat{x}')g_{,\beta}(\hat{x}') \\ &= (\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}])_{\alpha\beta} - \frac{1}{R'^2(\hat{x}')}\Lambda_{\alpha\rho}(\hat{x})f_{,\rho}(\hat{x})g_{,\beta}(\hat{x}') \\ &= (\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}])_{\alpha\beta}\end{aligned}$$

where use has been made of (2.7) together with $\Lambda_{\alpha\rho}(\hat{x})f_{,\rho}(\hat{x}) = 0$. □

3. *We can uniquely solve the second equation of (2.9) with respect to* $\hat{p}_\alpha$ *to obtain*

$$\hat{p}_\alpha = \tfrac{1}{2}\{(\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}])_{\alpha\beta}, \hat{p}'_\beta\}. \tag{2.13}$$

Proof. Taking the symmetrized products of $(\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}])_{\gamma\alpha}/2$ with both sides of the second equation of (2.9) we obtain with the help of (2.11) and (2.12)

$$\begin{aligned}\tfrac{1}{2}\{(\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}])_{\gamma\alpha}, \hat{p}'_\alpha\} &= \tfrac{1}{4}\{(\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}])_{\gamma\alpha}, \{(\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\alpha\beta}, \hat{p}_\beta\}\} \\ &= \tfrac{1}{2}\{(\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}]\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\gamma\beta}, \hat{p}_\beta\} \\ &= \tfrac{1}{2}\{(\Lambda(\hat{x})[\partial\hat{x}'/\partial\hat{x}][\partial\hat{x}/\partial\hat{x}'])_{\gamma\beta}, \hat{p}_\beta\} \\ &= \tfrac{1}{2}\{\Lambda_{\gamma\beta}(\hat{x}), \hat{p}_\beta\}\end{aligned}$$

where by virtue of (1.7), (2.11) and (1.9) the right-hand side reduces to

$$\begin{aligned}\hat{p}_\gamma - \frac{1}{2}\left\{\frac{f_{,\gamma}(\hat{x})f_{,\beta}(\hat{x})}{R^2(\hat{x})}, \hat{p}_\beta\right\} &= \hat{p}_\gamma - \frac{1}{4}\left\{\frac{f_{,\gamma}(\hat{x})}{R^2(\hat{x})}, \{f_{,\beta}(\hat{x}), \hat{p}_\beta\}\right\} \\ &= \hat{p}_\gamma.\end{aligned} \tag{2.14}$$

Thus we have proved (2.13). Uniqueness of the solution is evident from the process of calculation. □

Proof of theorem 1. To this end we first examine constraint (2.2) starting with the Dirac algebra on $f(x) = 0$. Taking the anti-symmetrized products of $g_{,\alpha}(\hat{x}')$ with both sides of the second equation of (2.9) we find

$$\begin{aligned}\{g_{,\alpha}(\hat{x}'), p'_\alpha\} &= \tfrac{1}{2}\{g_{,\alpha}(\hat{x}'), \{(\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\alpha\beta}, \hat{p}_\beta\}\} \\ &= \{g_{,\alpha}(\hat{x}')\Lambda'_{\alpha\gamma}(\hat{x}')[\partial\hat{x}/\partial\hat{x}']_{\gamma\beta}, \hat{p}_\beta\} = 0\end{aligned}$$

where use has been made of (2.11) and the identity $g_{,\alpha}(\hat{x}')\Lambda'_{\alpha\gamma}(\hat{x}') = 0$. Thus constraint (2.2) has been derived.

Next, to derive (2.4) we make a commutator of x'_α and p'_β. Then from (2.9) we obtain

$$\begin{aligned}[\hat{x}'_\alpha, \hat{p}'_\beta] &= \frac{1}{2}[\hat{x}'_\alpha, \{(\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}, \hat{p}_\gamma\}] \\ &= (\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}[\hat{x}'_\alpha, \hat{p}_\gamma] \\ &= \mathrm{i}\hbar(\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}\Lambda_{\gamma\rho}(\hat{x})[\partial\hat{x}'/\partial\hat{x}]_{\rho\alpha} \\ &= \mathrm{i}\hbar\Lambda'_{\alpha\beta}(\hat{x}') - \frac{\mathrm{i}\hbar}{R^2(\hat{x})}(\Lambda'(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}f_{,\gamma}(\hat{x})f_{,\rho}(\hat{x})[\partial\hat{x}'/\partial\hat{x}]_{\rho\alpha} \\ &= \mathrm{i}\hbar\Lambda'_{\alpha\beta}(\hat{x}') - \frac{\mathrm{i}\hbar}{R^2(\hat{x})}\Lambda'_{\beta\gamma}(\hat{x}')g_{,\gamma}(\hat{x}')f_{,\rho}(\hat{x})[\partial\hat{x}'/\partial x]_{\rho\alpha} \\ &= \mathrm{i}\hbar\Lambda'_{\alpha\beta}(\hat{x}')\end{aligned}$$

which proves (2.4).

Finally we will derive (2.5). To avoid unnecessary complications we will proceed in the following way: as seen from (1.11) and (1.12) the commutator $[\hat{p}'_\alpha, \hat{p}'_\beta]$ is linear in $\hat{p}_\gamma$, thereby applying (2.13) we can write it as

$$[\hat{p}'_\alpha, \hat{p}'_\beta] = \frac{\mathrm{i}\hbar}{2}\left\{c_\gamma^{[\alpha\beta]}(\hat{x}'), \hat{p}'_\gamma\right\} \tag{2.15}$$

with undetermined functions $c_\gamma^{[\alpha\beta]}(\hat{x}')$. Taking the commutators of $\hat{x}'_\gamma$ with both sides of (2.15) we obtain from the left-hand side

$$\begin{aligned}[\hat{x}'_\gamma, [\hat{p}'_\alpha, \hat{p}'_\beta]] &= [[\hat{x}'_\gamma, \hat{p}'_\alpha], \hat{p}'_\beta] + [\hat{p}'_\alpha, [\hat{x}'_\gamma, \hat{p}'_\beta]] \\ &= -\mathrm{i}\hbar\left[\frac{g_{,\gamma}(\hat{x}')g_{,\alpha}(\hat{x}')}{R'^2(\hat{x}')}, \hat{p}'_\beta\right] + \mathrm{i}\hbar\left[\frac{g_{,\gamma}(\hat{x}')g_{,\beta}(\hat{x}')}{R'^2(\hat{x}')}, \hat{p}'_\alpha\right]\end{aligned}$$

by virtue of (2.4), while from the right-hand side

$$\begin{aligned}\frac{\mathrm{i}\hbar}{2}\left\{c_\rho^{[\alpha\beta]}(\hat{x}'), [\hat{x}'_\gamma, \hat{p}'_\rho]\right\} &= -\hbar^2\Lambda'_{\gamma\rho}(\hat{x}')c_\rho^{[\alpha\beta]}(\hat{x}') \\ &= -\hbar^2 c_\gamma^{[\alpha\beta]}(\hat{x}') + \frac{\hbar^2}{R'^2(\hat{x}')}g_{,\gamma}(\hat{x}')g_{,\rho}(\hat{x}')c_\rho^{[\alpha\beta]}(\hat{x}').\end{aligned}$$

Since the right-hand sides of the above two equations must be the same we find

$$\begin{aligned}\hbar c_\gamma^{[\alpha\beta]}(\hat{x}') = {}& \frac{\hbar}{R'^2(\hat{x}')}g_{,\gamma}(\hat{x}')g_{,\rho}(\hat{x}')c_\rho^{[\alpha\beta]}(\hat{x}') \\ &+ \mathrm{i}\left[\frac{g_{,\gamma}(\hat{x}')g_{,\alpha}(\hat{x}')}{R'^2(\hat{x}')}, \hat{p}'_\beta\right] - \mathrm{i}\left[\frac{g_{,\gamma}(\hat{x}')g_{,\beta}(\hat{x}')}{R'^2(\hat{x}')}, \hat{p}'_\alpha\right].\end{aligned}$$

Then inserting this relation into the right-hand side of (2.15) we find

$$\begin{aligned}[\hat{p}'_\alpha, \hat{p}'_\beta] = {}& \frac{\mathrm{i}\hbar}{2}\left\{\frac{1}{R'^2(\hat{x}')}c_\rho^{[\alpha\beta]}(\hat{x}')g_{,\rho}(\hat{x}')g_{,\gamma}(\hat{x}'), \hat{p}'_\gamma\right\} \\ &- \frac{1}{2}\left(\left\{\left[\frac{1}{R'^2(\hat{x}')}g_{,\alpha}(\hat{x}')g_{,\gamma}(\hat{x}'), \hat{p}'_\beta\right], \hat{p}'_\gamma\right\} - (\alpha \leftrightarrow \beta)\right)\end{aligned}$$

where the first term of the right-hand side is found to vanish owing to (2.11) and (2.2). Hence we can write it as

$$[\hat{p}'_\alpha, \hat{p}'_\beta] = -\frac{1}{2}\left(\left\{\left[\frac{1}{R'^2(\hat{x}')}g_{,\alpha}(\hat{x}')g_{,\gamma}(\hat{x}'), \hat{p}'_\beta\right], \hat{p}'_\gamma\right\} - (\alpha \leftrightarrow \beta)\right). \tag{2.16}$$

On the other hand, with the aid of (2.4) we have by direct calculation

$$\begin{aligned}\left[\frac{1}{R'^2(\hat{x}')}g_{,\alpha}(\hat{x}')g_{,\gamma}(\hat{x}'), \hat{p}'_\beta\right] &= \mathrm{i}\hbar\Lambda'_{\rho\beta}(\hat{x}')\frac{\partial}{\partial\hat{x}'_\rho}\left(\frac{1}{R'^2(\hat{x}')}g_{,\alpha}(\hat{x}')g_{,\gamma}(\hat{x}')\right) \\ = \frac{\mathrm{i}\hbar}{R'^2(\hat{x}')}&\left(g_{,\alpha\beta}(\hat{x}')g_{,\gamma}(\hat{x}') - g_{,\alpha}(\hat{x}')g_{,\beta}(\hat{x}')g_{,\rho}(\hat{x}')g_{,\gamma}(\hat{x}')\frac{\partial}{\partial\hat{x}'_\rho}\left(\frac{1}{R'^2(\hat{x}')}\right)\right. \\ &\left.- \frac{g_{,\alpha}(\hat{x}')g_{,\beta}(\hat{x}')g_{,\rho}(\hat{x}')g_{,\gamma\rho}(\hat{x}')}{R'^2(\hat{x}')}\right) + \frac{\mathrm{i}\hbar}{R'^2(\hat{x}')}g_{,\alpha}(\hat{x}')g_{,\beta\gamma}(\hat{x}') \\ &- \frac{\mathrm{i}\hbar}{R'^4(\hat{x}')}(2g_{,\beta\rho}(\hat{x}')g_{,\alpha}(\hat{x}')g_{,\rho}(\hat{x}') + g_{,\alpha\rho}(x')g_{,\rho}(\hat{x}')g_{,\beta}(\hat{x}'))g_{,\gamma}(\hat{x}')\end{aligned}$$

which immediately leads to

$$\left[\frac{1}{R'^2(\hat{x}')}g_{,\alpha}(\hat{x}')g_{,\gamma}(\hat{x}'),\,\hat{p}'_\beta\right]-(\alpha\leftrightarrow\beta)$$
$$=\frac{\mathrm{i}\hbar}{R'^2(\hat{x}')}(g_{,\alpha}(\hat{x}')g_{,\beta\gamma}(\hat{x}')-g_{,\beta}(\hat{x}')g_{,\alpha\gamma}(\hat{x}'))$$
$$-\frac{\mathrm{i}\hbar}{R'^4(\hat{x}')}(g_{,\alpha}(\hat{x}')g_{,\beta\rho}(\hat{x}')g_{,\rho}(\hat{x}')-g_{,\beta}(\hat{x}')g_{,\alpha\rho}(\hat{x}')g_{,\rho}(\hat{x}'))g_{,\gamma}(\hat{x}'). \tag{2.17}$$

Then taking the anti-commutators of $\hat{p}'_\gamma$ with both sides of this equation we find that the contribution from the second term of the right-hand side becomes zero due to (2.11) and (2.2), and finally obtain from (2.16)

$$[\hat{p}'_\alpha,\hat{p}'_\beta]=-\frac{\mathrm{i}\hbar}{2}\left\{\frac{1}{R'^2(\hat{x}')}(g_{,\alpha}(\hat{x}')g_{,\beta\gamma}(\hat{x}')-g_{,\beta}(\hat{x}')g_{,\alpha\gamma}(\hat{x}')),\,\hat{p}'_\gamma\right\}$$

thereby proving (2.5).

Thus we have completed the proof of theorem 1. □

Since, as was mentioned already, transformation (2.9) is invertible, the two descriptions based on the respective Dirac algebras on $f(x)=0$ and on $g(x)=0$ are seen to be equivalent. Thus, if conversely starting with the canonical variables $\hat{x}_\alpha$ and $\hat{p}_\alpha$ that satisfy the Dirac algebra on $g(x)=0$ we then obtain those of the Dirac algebra on $f(x)=0$. They can be written as

$$\begin{cases}\hat{x}'_\alpha=x'_\alpha(\hat{x})\\ \hat{p}'_\alpha=\frac{1}{2}\{(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\alpha\beta},\,\hat{p}_\beta\}\end{cases} \tag{2.18}$$

where the first line stands for the diffeomorphism from the manifold $g(x)=0$ to $f(x)=0$, and hence it satisfies $f(x')=g(x)$. Since (2.18) is uniquely given by (2.9) there holds a one-to-one correspondence between representations of the two Dirac algebras. Thus if one of them is irreducible then so is the other as well. Furthermore, it is also remarkable that the irreducible representation space of $(\hat{x}_\alpha,\hat{p}_\alpha)$ is the same as that of $(\hat{x}'_\alpha,\hat{p}'_\alpha)$, i.e., in our case the irreducible representation space of Dirac algebra is kept unchanged under smooth deformations of the manifold. Based on these facts we will obtain, in the next section, all possible irreducible representations of the Dirac algebra on $f(x)=0$.

3. Irreducible representations

To determine irreducible representations of Dirac algebra on $f(x)=0$ we will use S^D for the manifold $g(x)=0$ in (2.18), since the irreducible representations in the case of S^D are known completely [2].

Thus to begin with we briefly survey the irreducible representations of the Dirac algebra on S^D with radius r. Then the algebra is given by

$$g(\hat{x})\equiv\hat{x}^2-r^2=0 \tag{3.1}$$

$$\{\hat{x}_\alpha,\hat{p}_\alpha\}=0 \tag{3.2}$$

$$[\hat{x}_\alpha,\hat{x}_\beta]=0 \tag{3.3}$$

$$[\hat{x}_\alpha,\hat{p}_\beta]=\mathrm{i}\hbar\lambda_{\alpha\beta}(\hat{x}) \tag{3.4}$$

$$[\hat{p}_\alpha,\hat{p}_\beta]=-\mathrm{i}\hbar\left\{\frac{1}{\hat{x}^2}(\delta_{\beta\gamma}\hat{x}_\alpha-\delta_{\alpha\gamma}\hat{x}_\beta),\,\hat{p}_\gamma\right\} \tag{3.5}$$

where

$$\lambda_{\alpha\beta}(\hat{x}) = \delta_{\alpha\beta} - \frac{\hat{x}_\alpha \hat{x}_\beta}{\hat{x}^2}. \tag{3.6}$$

In this connection it was shown [2] that the operators $\hat{p}_\beta$ in any irreducible representation are written in terms of $\hat{x}_\alpha$ and $\hat{\pi}_\alpha$, which obey the canonical commutation relations in $(D+1)$ dimensions

$$\begin{aligned} &[\hat{x}_\alpha, \hat{x}_\beta] = [\hat{\pi}_\alpha, \hat{\pi}_\beta] = 0 \\ &[\hat{x}_\alpha, \hat{\pi}_\beta] = \mathrm{i}\hbar\delta_{\alpha\beta}. \end{aligned} \tag{3.7}$$

They are represented as follows:

$\boldsymbol{D = 1}$

$$\hat{p}_\beta = \frac{1}{2}\left\{\frac{\hat{x}_\rho}{\hat{x}^2}, \hat{L}_{\rho\beta}\right\} - \alpha\hbar\frac{\epsilon_{\beta\rho}\hat{x}_\rho}{\hat{x}^2} \qquad (0 \leqslant \alpha < 1) \tag{3.8}$$

where

$$\hat{L}_{\rho\beta} = -\hat{L}_{\beta\rho} \qquad \hat{L}_{12} = \hat{x}_1\hat{\pi}_2 - \hat{x}_2\hat{\pi}_1 \qquad \text{and} \qquad \epsilon_{\beta\gamma} = -\epsilon_{\gamma\beta} \quad \epsilon_{12} = 1.$$

Any irreducible representation in this case can be uniquely specified by the parameter α.

$\boldsymbol{D \geqslant 2}$

$$\hat{p}_\beta = \frac{1}{2}\left\{\frac{\hat{x}_\rho}{\hat{x}^2}, \hat{L}_{\rho\beta}\right\} \qquad (\beta = 1, 2, 3, \ldots, D+1) \tag{3.9}$$

where

$$\hat{L}_{\rho\beta} = \hat{x}_\rho\hat{\pi}_\beta - \hat{x}_\beta\hat{\pi}_\rho.$$

In this case, for each D the irreducible representation is uniquely determined except for unitary-equivalent representations.

Thus by inserting (3.8) or (3.9) into the right-hand side of the second equation of (2.18) we are able to derive $\hat{p}'_\beta$ in the irreducible representation for the Dirac algebra on $f(x') = 0$.

To this end we first consider the case of $D \geqslant 2$ for simplicity. Then from (3.9) and (2.18) we obtain

$$\begin{aligned} \hat{p}'_\beta &= \frac{1}{4}\left\{(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}, \left\{\frac{\hat{x}_\rho}{\hat{x}^2}, \hat{L}_{\rho\gamma}\right\}\right\} \\ &= \frac{1}{4}\{(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}, \{\lambda_{\gamma\rho}(\hat{x}), \hat{\pi}_\rho\}\} \\ &= \frac{1}{2}\{(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}']\lambda(\hat{x}))_{\beta\gamma}, \hat{\pi}_\gamma\} \end{aligned} \tag{3.10}$$

where $x'_\alpha = x'_\alpha(x)$ is a diffeomorphism satisfying $f(x') = x^2 - r^2$. Note here that as seen from (2.12) there holds the identity

$$\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}']\lambda(\hat{x}) = \Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'].$$

Hence, $\hat{p}'_\beta$ of (3.10) can be written as

$$\begin{aligned} \hat{p}'_\beta &= \tfrac{1}{2}\{(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}, \hat{\pi}_\gamma\} \\ &= \tfrac{1}{4}\{\Lambda_{\beta\rho}(\hat{x}'), \{[\partial\hat{x}/\partial\hat{x}']_{\rho\gamma}, \hat{\pi}_\gamma\}\} \\ &= \tfrac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}'), \hat{\pi}'_\gamma\} \end{aligned} \tag{3.11}$$

with

$$\hat{\pi}'_\alpha = \tfrac{1}{2}\{[\partial\hat{x}/\partial\hat{x}']_{\alpha\beta}, \hat{\pi}_\beta\}. \tag{3.12}$$

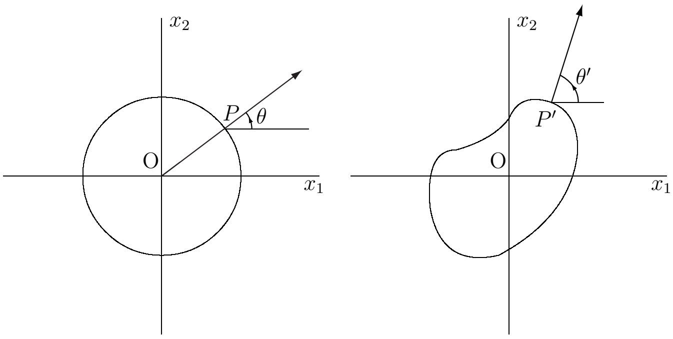

Figure 1. $P = (x_1, x_2)$, $P' = (x'_1, x'_2)$, where $x'_\alpha = x'_\alpha(x)$ $(\alpha = 1, 2)$.

Furthermore, it can be shown [3] that the operators $\hat{\pi}'_\alpha$ are self-adjoint on the representation space of (3.7) and obey the canonical commutation relations

$$[\hat{x}'_\alpha, \hat{x}'_\beta] = [\hat{\pi}'_\alpha, \hat{\pi}'_\beta] = 0$$

$$[\hat{x}'_\alpha, \hat{\pi}'_\beta] = \mathrm{i}\hbar\delta_{\alpha\beta}.$$

Thus removing all primes from (3.11) we obtain

$$\hat{p}_\beta = \tfrac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}), \hat{\pi}_\gamma\} \qquad \text{for} \quad D \geqslant 2 \tag{3.13}$$

which of course realize a unique irreducible representation of the Dirac algebra on $f(x) = 0$ together with $\hat{x}_\alpha$ for each D ($\geqslant$2).

Next let us consider the case of $D = 1$. Inserting (3.8) into the second equation of (2.18) and applying the same procedure as above we find

$$\begin{aligned}\hat{p}'_\beta &= \frac{1}{4}\left\{(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}, \left\{\frac{\hat{x}_\rho}{\hat{x}^2}, \hat{L}_{\rho\gamma}\right\}\right\} - \frac{\alpha\hbar}{\hat{x}^2}(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}\hat{x}_\rho\epsilon_{\gamma\rho} \\ &= \frac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}'), \hat{\pi}'_\gamma\} - \frac{\alpha\hbar}{\hat{x}^2}(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}\hat{x}_\rho\epsilon_{\gamma\rho}.\end{aligned} \tag{3.14}$$

The right-hand side contains the unknown matrix $[\partial\hat{x}/\partial\hat{x}']$ which depends on the choice of diffeomorphism from S^D to $f(x) = 0$. Since in our case the diffeomorphism is not unique, we have to eliminate $[\partial\hat{x}/\partial\hat{x}']$ from the expression of $\hat{p}'_\beta$ to derive a definite form. For this purpose we introduce an angle θ such that

$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right). \tag{3.15}$$

Then there holds

$$-\frac{x_\rho\epsilon_{\gamma\rho}}{x^2} = \frac{\partial\theta}{\partial x_\gamma} \tag{3.16}$$

which leads us to

$$-\frac{\alpha\hbar}{\hat{x}^2}(\Lambda(\hat{x}')[\partial\hat{x}/\partial\hat{x}'])_{\beta\gamma}\hat{x}_\rho\epsilon_{\gamma\rho} = \alpha\hbar\Lambda_{\beta\rho}(\hat{x}')\widehat{\left(\frac{\partial\theta}{\partial x'_\rho}\right)}. \tag{3.17}$$

Corresponding to the angle θ we further introduce θ' which denotes the angle between the x_1-axis on $\mathbb{R}^2$ and a normal to the closed curve $f(x') = 0$ at the point $P' = (x'_1, x'_2)$ (see figure 1). It is given by

$$\theta' = \tan^{-1}\left(\frac{f_{,2}(x')}{f_{,1}(x')}\right). \tag{3.18}$$

Though both θ and θ' are multivalued functions of x (and hence of x'), the difference $\xi(x') = \theta' - \theta$ provides us with a single-valued function of x'. In fact, from figure 1 it is observed that as the angle θ is transformed as $\theta \rightarrow \theta + 2\pi$ the angle θ' transforms as $\theta' \rightarrow \theta' + 2\pi$, and vice versa. Hence in contrast to θ and θ' the quantity $\xi(x')$ can have an operator form $\xi(\hat{x}')$. Thus applying the unitary operator

$$u(\hat{x}') \equiv \exp[\mathrm{i}\alpha\xi(\hat{x}')] \tag{3.19}$$

we obtain from (3.14) and (3.16)

$$\begin{aligned}
u^{\dagger}(\hat{x}')\hat{p}'_{\beta}u(\hat{x}') &= \frac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}'), \hat{\pi}'_{\gamma}\} + \mathrm{i}\alpha\Lambda_{\beta\gamma}(\hat{x}')[\hat{\pi}'_{\gamma}, \xi(\hat{x}')] + \alpha\hbar\Lambda_{\beta\gamma}(\hat{x}')\widehat{\left(\frac{\partial\theta}{\partial x'_{\gamma}}\right)} \\
&= \frac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}'), \hat{\pi}'_{\gamma}\} + \alpha\hbar\Lambda_{\beta\gamma}(\hat{x}')\widehat{\left(\frac{\partial\theta'}{\partial x'_{\gamma}}\right)} \\
&= \frac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}'), \hat{\pi}'_{\gamma}\} + \alpha\hbar\Lambda_{\beta\gamma}(\hat{x}')\frac{f_{,2\gamma}(\hat{x}')f_{,1}(\hat{x}') - f_{,1\gamma}(\hat{x}')f_{,2}(\hat{x}')}{R^2(\hat{x}')} \\
&= \frac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}'), \hat{\pi}'_{\gamma}\} - \alpha\hbar\frac{\Lambda_{\beta\gamma}(\hat{x}')f_{,\gamma\rho}(\hat{x}')f_{,\sigma}(\hat{x}')\epsilon_{\rho\sigma}}{R^2(\hat{x}')}.
\end{aligned} \tag{3.20}$$

The right-hand side is unitary-equivalent to (3.14) which, given a parameter $\alpha \in [0, 1)$, is known to provide an irreducible representation of the Dirac algebra on $f(\hat{x}') = 0$. Then removing all primes in (3.20) we have the expression for $\hat{p}_{\beta}$ in the irreducible representation of the Dirac algebra for $D = 1$.

Thus summarizing the above, we now arrive at the following:

Theorem 2. *The operators $\hat{p}_{\beta}$, which together with $\hat{x}_{\alpha}$ describe the irreducible representation of the Dirac algebra on $f(x) = 0$, are uniquely given by*

$$\hat{p}_{\beta} = \frac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}), \hat{\pi}_{\gamma}\} - \alpha\hbar\frac{\Lambda_{\beta\gamma}(\hat{x})f_{,\gamma\rho}(\hat{x})f_{,\sigma}(\hat{x})\epsilon_{\rho\sigma}}{R^2(\hat{x})} \qquad \textit{for} \quad D = 1 \qquad (0 \leqslant \alpha < 1) \tag{3.21}$$

and

$$\hat{p}_{\beta} = \tfrac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}), \hat{\pi}_{\gamma}\} \qquad \textit{for} \quad D \geqslant 2 \tag{3.22}$$

where the operators $\hat{x}_{\alpha}$ and $\hat{\pi}_{\alpha}$ are assumed to obey the canonical commutation relations given by (3.7).

All possible irreducible representations of the Dirac algebra on $f(x) = 0$ are exhausted by (3.21) and (3.22) except for similarity-equivalent representations.

By direct calculation it is also seen that the operators $\hat{p}_{\beta}$ thus obtained satisfy (1.9)–(1.12) on the basis of canonical commutation relations (3.7) without the use of the primary constraint (1.8). Accordingly to describe the irreducible representation it is sufficient for us to apply the expression (3.21) for $D = 1$ or (3.22) for $D \geqslant 2$ in addition to the primary constraint. In other words, equations (1.9)–(1.12) are no longer necessary, for they are already involved in (3.21) (or (3.22)). Hence instead of them we can utilize the well-known relations (3.7).

The Hermitian property of $\hat{p}_{\beta}$ on the irreducible representation space will be discussed in the next section.

4. Inner products and $\hat{x}$-diagonal representation

Now given an irreducible representation we denote the corresponding representation space as $\underline{\mathcal{H}}$ and state vectors belonging to it as $|\underline{\psi}), |\underline{\chi}), \ldots$. We further write a point on $f(x) = 0$ as $\underline{x}$. Since the eigenstates $|\underline{x})$ of the position operators on the manifold form an ortho-complete system in $\underline{\mathcal{H}}$ we may write the completeness condition as

$$\int_{\Sigma_f} \mathrm{d}^D\sigma(\underline{x})\, |\underline{x})(\underline{x}| = \underline{\hat{1}} \tag{4.1}$$

where $\underline{\hat{1}}$ is the unit operator on $\underline{\mathcal{H}}$. The infinitesimal volume element $\mathrm{d}^D\sigma(\underline{x})$ of the manifold is defined referring to the measure of the flat space $\mathbb{R}^{D+1}$, and Σ_f denotes the whole domain of the manifold. In the position diagonal representation we write the wavefunctions corresponding to $|\underline{\psi})$ and $|\underline{\chi})$ as $\underline{\psi}(\underline{x}) \equiv (\underline{x}|\underline{\psi})$ and $\underline{\chi}(\underline{x}) \equiv (\underline{x}|\underline{\chi})$, respectively.

On the other hand, we will denote the representation space of the canonical commutation relations (3.7) as $\mathcal{H}$, which is spanned by the eigenstates $|x\rangle$ of the operators $\hat{x}_\beta$. They of course obey

$$\hat{x}_\beta|x\rangle = x_\beta|x\rangle \qquad \langle x|x'\rangle = \delta^{D+1}(x - x') \tag{4.2}$$

$$\int \mathrm{d}^{D+1}x\, |x\rangle\langle x| = \hat{1} \tag{4.3}$$

with $\hat{1}$ being the unit operator on $\mathcal{H}$. Corresponding to $\underline{\psi}(\underline{x})$ and $\underline{\chi}(\underline{x})$ we introduce auxiliary 'wavefunctions' $\psi(x) \equiv \langle x|\psi\rangle$ and $\chi(x) \equiv \langle x|\chi\rangle$ for $|\psi\rangle, |\chi\rangle \in \mathcal{H}$. They are assumed to satisfy the conditions

$$\underline{\psi}(\underline{x}) = \psi(x)|_{x=\underline{x}} \qquad \underline{\chi}(\underline{x}) = \chi(x)|_{x=\underline{x}}. \tag{4.4}$$

Then we have

$$(\underline{\psi}, \underline{\chi}) = \int_{\Sigma_f} \mathrm{d}^D\sigma(\underline{x})\, \underline{\psi}^*(\underline{x})\underline{\chi}(\underline{x}) = \int \mathrm{d}^{D+1}x\, \delta(f(x))R(x)\psi^*(x)\chi(x) \tag{4.5}$$

where use has been made of the relation $f(x) = (x - \underline{x})_\alpha f_{,\alpha}(\underline{x})$ for small $|x - \underline{x}|$.

In this connection we note that in the Dirac algebra on $f(x) = 0$ there exists an ambiguity of multiplying $f(x)$ by a non-vanishing factor to describe the same manifold. In fact it is shown with some calculations that under the transformation $f(x) \to f(x)C(x)$ $(C(x) \neq 0)$ the Dirac algebra (1.8)–(1.12) remains unchanged and hence so is the expression for $\hat{p}_\beta$ in theorem 2. Utilizing this we can always convert $f(x)$ into a *normalized* form that is defined to satisfy

$$R^2(\underline{x}) = f_{,\alpha}(\underline{x})f_{,\alpha}(\underline{x}) = 1 \qquad \forall\underline{x} \text{ on } f(x) = 0 \tag{4.6}$$

with which the inner product is reduced to

$$(\underline{\psi}, \underline{\chi}) = \int \mathrm{d}^{D+1}x\, \delta(f(x))\psi^*(x)\chi(x). \tag{4.7}$$

As will be seen soon, condition (4.6) plays a definite role in guaranteeing the Hermiticity of $\hat{p}_\beta$ stated in theorem 2.

Since $|\underline{\psi})$ and $|\psi\rangle$ are related only through (4.4), the correspondence of the latter to the former is not unique. In fact, $\mathcal{H}$ and $\underline{\mathcal{H}}$ are thought to be the total and base spaces respectively in the theory of fibre space. Each equation of (4.4) implies the projection of a point on a fibre to the base space. Thus the correspondence may be written as

$$\Pi \cdot |\psi\rangle = |\underline{\psi}) \tag{4.8}$$

with the aid of projection Π. Needless to say, it has nothing to do with the usual projection operator on a Hilbert space because its domain and range are specified by different spaces described by $\mathcal{H}$ and $\underline{\mathcal{H}}$ respectively.

It is noted that an arbitrary operator $O(\hat{x}, \hat{p})$ on $\underline{\mathcal{H}}$ can also be regarded as an operator on $\mathcal{H}$. Since owing to (4.4) it is commutable with Π, we find

$$\Pi \cdot O(\hat{x}, \hat{p})|\psi\rangle = O(\hat{x}, \hat{p})|\underline{\psi}) \tag{4.9}$$

and hence

$$\begin{aligned}(\underline{\psi}|O(\hat{x}, \hat{p})|\underline{\chi}) &= \int_{\Sigma_f} \mathrm{d}^D\sigma(\underline{x})\,(\underline{\psi}|\underline{x})(\underline{x}|O(\hat{x}, \hat{p})|\underline{\chi}) \\ &= \int \mathrm{d}^{D+1}x\,\delta(f(x))\langle\psi|x\rangle\langle x|O(\hat{x}, \hat{p})|\chi\rangle \end{aligned}\tag{4.10}$$

which enables us to write

$$(\underline{\psi}|O(\hat{x}, \hat{p})|\underline{\chi}) = \int \mathrm{d}^{D+1}x\,\mathrm{d}^{D+1}x'\,\delta(f(x))\langle x|O(\hat{x}, \hat{p})|x'\rangle\psi^*(x)\chi(x'). \tag{4.11}$$

Thus any matrix element of $O(\hat{x}, \hat{p})$ on $\underline{\mathcal{H}}$ is obtainable by calculating the quantity $\langle x|O(\hat{x}, \hat{p})|x'\rangle$ on $\mathcal{H}$. This will provide us with a powerful tool, e.g., for rigorous formulation of the path integral on $f(x) = 0$ [4].

Now let us examine the Hermiticity of $\hat{p}_\alpha$ on $\underline{\mathcal{H}}$. Since they are Hermitian on $\mathcal{H}$, we immediately obtain from (4.7)

$$\begin{aligned}\int \mathrm{d}^{D+1}x\,\delta(f(x))\psi^*(x)\{\hat{p}_\alpha\chi(x)\} &= \int \mathrm{d}^{D+1}x\,\{\hat{p}_\alpha\delta(f(x))\psi(x)\}^*\chi(x) \\ &= \int \mathrm{d}^{D+1}x\,\delta(f(x))\{\hat{p}_\alpha\psi(x)\}^*\chi(x)\end{aligned}\tag{4.12}$$

which implies the Hermiticity of $\hat{p}_\alpha$ on the physical Hilbert space $\underline{\mathcal{H}}$;

$$(\underline{\psi}|\hat{p}_\alpha|\underline{\chi})^* = (\underline{\chi}|\hat{p}_\alpha|\underline{\psi}) \tag{4.13}$$

where condition (4.6) has played a crucial role. In fact, if starting with the general case (4.5), we observe that a necessary and sufficient condition for $\hat{p}_\beta$ in theorem 2 to be Hermitian is $[\hat{p}_\beta, \delta(f(\hat{x}))R(\hat{x})] = 0$, and hence we have $\delta(f(x))R(\hat{x}) = c\delta(f(x))$ with c being an arbitrary positive constant. In (4.6) we put $c = 1$ for simplicity.

Before closing this section we have to note that operators $\hat{p}_\beta$ that satisfy the Dirac algebra on $f(x) = 0$ were found inductively by Kashiwa [5] through an elaborate analysis of the Faddeev–Senjanovic formula [6, 7] for path integrals. In our notation the expression obtained by him may be written as

$$\hat{p}^{\mathrm{K}}_\alpha = \Lambda_{\alpha\beta}(\hat{x})\hat{\pi}_\beta - \frac{\mathrm{i}}{2}\partial_\beta\Lambda_{\alpha\beta}(\hat{x}) - \frac{\mathrm{i}}{2}\Lambda_{\alpha\beta}(\hat{x})\partial_\gamma\Lambda_{\beta\gamma}(\hat{x}) \tag{4.14}$$

in units of $\hbar = 1$. Since the combination of the first and second terms in the right-hand side is $\{\Lambda_{\alpha\beta}(\hat{x}), \hat{\pi}_\beta\}/2$, expression (4.14) is not Hermitian. Thus owing to theorem 2 there must exist a similarity transformation eliminating the third term. To this end we apply the relation

$$\begin{aligned}-\partial_\gamma\Lambda_{\beta\gamma}(\hat{x}) &= f_{,\beta}(\hat{x})\partial_\gamma\left(\frac{f_{,\gamma}(\hat{x})}{R^2(\hat{x})}\right) + \frac{f_{,\beta\gamma}(\hat{x})f_{,\gamma}(\hat{x})}{R^2(\hat{x})} \\ &= f_{,\beta}(\hat{x})\partial_\gamma\left(\frac{f_{,\gamma}(\hat{x})}{R^2(\hat{x})}\right) + \frac{1}{2}\partial_\beta\log R^2(\hat{x})\end{aligned}\tag{4.15}$$

to the third term. Then we obtain

$$\begin{aligned}\hat{p}^{\mathrm{K}}_\alpha &= \frac{1}{2}\{\Lambda_{\alpha\beta}(\hat{x}), \hat{\pi}_\beta\} + \frac{\mathrm{i}}{4}\Lambda_{\alpha\beta}(\hat{x})\partial_\beta\log R^2(\hat{x}) \\ &= \frac{1}{2}V(\hat{x})\{\Lambda_{\alpha\beta}(\hat{x}), \hat{\pi}_\beta\}V^{-1}(\hat{x})\end{aligned}\tag{4.16}$$

with

$$V(\hat{x}) = \exp\left[\tfrac{1}{4}\log R^2(\hat{x})\right] = \sqrt{R(\hat{x})}.$$

Equation (4.16) demonstrates a similarity equivalence of $\hat{p}^{\mathrm{K}}_{\alpha}$ to (3.21) with $\alpha = 0$ for $D = 1$ and to (3.22) for $D \geqslant 2$.

5. Concluding remarks

In the present paper we have determined all possible irreducible representations of the Dirac algebra on $f(x) = 0$ by applying theorem 1. It has enabled us to establish a connection between two Dirac algebras defined respectively on two manifolds, each of which is diffeomorphic to S^D. In so doing the diffeomorphism has been given by a kind of variable transformation keeping state vectors unchanged. Thus at least in our case the irreducible representation space of the algebra depends only on the topology of the manifold and not on its shape.

We have also derived the explicit form of $\hat{p}_\beta$ in an arbitrary irreducible representation of the Dirac algebra on $f(x) = 0$. The result is stated as theorem 2. Moreover, Hermiticity of $\hat{p}_\beta$ in (3.21) and (3.22) has been shown with the aid of $f(x)$ that obeys the normalization condition (4.6). Thus it may be emphasized that the use of normalized $f(x)$ in our formalism is inevitable from the physical point of view.

Related to this it is conjectured that $\hat{p}_\beta$ are presumably self-adjoint on $\underline{\mathcal{H}}$ and each of them takes a continuous spectrum of range $(-\infty, \infty)$. In the case for S^D this was explicitly shown [2] by solving the eigenvalue problem for $\hat{p}_\beta$. In general cases, however, the situation seems not very simple. We have not yet succeeded in giving an answer to this problem. In this respect it would be worthwhile to give a proof of the self-adjointness of the operator $\hat{p}_\beta$ mentioned in theorem 2.

One application of our result would be to derive a rigorous expression for the path integral based on the irreducible representation of the Dirac algebra. A detailed study on the path integral will be made in a forthcoming paper [4].

It is noted that a few years ago essentially the same results as in the present paper were found by the author [8] with successive applications of infinitesimal deformation of the manifold in obtaining $f(x) = 0$ from S^D. They consist of two kinds of diffeomorphism, that is, dilatation and area-preserving mapping [9], the latter of which is defined to preserve the measure $\mathrm{d}^D\sigma(\underline{x})$. However, the method employed therein was tedious and incomplete in some respects. Hence in the present paper we have developed a simpler and well-defined approach without recourse to infinitesimal deformations of the manifold. In spite of this, such a manifold deformation technique would be important from a geometrical point of view, especially in relation to the argument by Karasev and Maslov [10], though this is not an operator formalism. Thus a further study of this problem would be highly desirable.

The formalism given in the present paper can easily be generalized to the case where the manifold $f(x) = 0$ is diffeomorphic to $\mathbb{R}^D$. The Dirac algebra in this case takes the same form as (1.8)–(1.12), and the irreducible $\hat{p}_\beta$ are given by (3.21) and (3.22). For $D = 1$, however, it is noted that $\theta'(x) \equiv \tan^{-1}(f_{,2}(x)/f_{,1}(x))$ is a single-valued function of x in the neighbourhood of $f(x) = 0$ because of the simply connected structure of the manifold. Thus by means of the unitary operator $\mathrm{e}^{\mathrm{i}\alpha\theta'(\hat{x})}$ we can eliminate the parameter α from the right-hand side of (3.21) to obtain

$$\mathrm{e}^{\mathrm{i}\alpha\theta'(\hat{x})}[\text{rhs of (3.21)}]\,\mathrm{e}^{-\mathrm{i}\alpha\theta'(\hat{x})} = \tfrac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}), \hat{\pi}_\gamma\}. \tag{5.1}$$

Consequently, if $f(x) = 0$ is diffeomorphic to $\mathbb{R}^D$, the irreducible $\hat{p}_\beta$ is uniquely given by

$$\hat{p}_\beta = \tfrac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}), \hat{\pi}_\gamma\} \tag{5.2}$$

for any D.

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